JOURNAL OF COMBINATORIAL THEORY 9, 324–326 (1970)

On Nonreconstructable Tournaments*

L. W. BEINEKE AND E. T. PARKER

Purdue University at Fort Wayne, Indiana 46805, and University of Illinois, Urbana, Illinois 61801

Communicated by Frank Harary

Received December 7, 1968

ABSTRACT

Pairs of non-isomorphic strong tournaments of orders 5 and 6 are given for which the subtournaments of orders 4 and 5, respectively, are pairwise isomorphic. Heretofore, only pairs of orders 3 and 4 were known.

A *tournament* of order n has n nodes with exactly one directed arc joining each pair of distinct nodes; in other words, it is a complete, irreflexive, asymmetric relation on a set of n elements. A tournament T of order n clearly determines n subtournaments of order $n - 1$, each obtained from T by deleting one node and the incident arcs. A problem in the study of tournaments, sometimes called Ulam's problem in analogy with one in undirected graphs, has been considered by Harary and Palmer [1] and can be stated as follows: For what pairs of non-isomorphic tournaments of order n are the two sets of n subtournaments of order $n - 1$ pairwise-isomorphic ? Alternatively, given n tournaments of order $n - 1$, when can non-isomorphic tournaments of order n be reconstructed from them ?

A tournament is called *strong* if there is no partition of its nodes into non-empty subsets U and V such that all arcs go from U to V. A well-known theorem states that a tournament is strong if and only if it has a spanning directed cycle. Harary and Palmer showed that, if two non-isomorphic tournaments of order $n > 4$ are not strong, then their subtournaments of order $n - 1$ cannot be pairwise-isomorphic. It should be noted (see Moon [2, p. 3]) that "almost all" tournaments are strong, however.

* Research of first author supported by Air Force Office of Scientific Research grant AFOSR-68-1515. Research of second author supported by Office of Naval Research grant N000 14-67-A-0305-0008.

The phenomenon of pairs of tournaments of order n having their subtournaments of order $n - 1$ pairwise-isomorphic is apparently quite rare. The example of order 3, shown in Figure 1, must be regarded as a degenerate case, since there is only one tournament of order 2. However, the example of order 4, given in [1] and shown in Figure 2, cannot be considered degenerate.

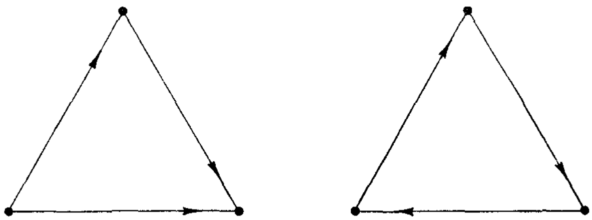

FIG. 1. Order 3.

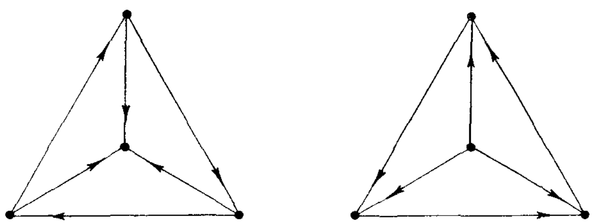

FIG. 2. Order 4.

Harary and Palmer conjectured the non-existence of examples of higher order, but we give examples of orders 5 and 6 here. Schematically, these are shown in Figures 3 and 4, in which the arcs not shown are understood to be directed from left to right. (This type of diagram was used by Moon [2] in his book on tournaments.)

FIG. 3. Order 5.

FIG. 4. Order 6.

To see that the tournaments of order 5 are non-isomorphic, we note that, although the list of scores (numbers of out-going arcs from the nodes) is (3, 2, 2, 2, 1) in both, in one the arc from the node of score 1 goes to the one of score 3, while in the other the corresponding arc goes to a node of score 2. Since each of the two tournaments of order 6 has a unique transitive subtournament of order 5, they are clearly non-isomorphic.

There are no known larger examples; perhaps there are none. Or is there an example for every order ? Such conjectures and questions abound. By considering the converse tournaments (obtained by reversing the direction of every arc), we observe that the pairs of tournaments of orders 4 and 6 are converses of each other, while the tournaments of orders 3 and 5 are self-converse. Is this a pattern which might prove helpful ? In conclusion we mention the analogous problem for directed graphs in general: Do examples exist which are not tournaments ?

Added in proof. Exhaustive examination of all 56 tournaments with six nodes has yielded two additional pairs whose subtournaments of order 5 are pairwise-isomorphic. These are shown in Figs. 5 and 6. In Fig. 5, the arc from the node with score 1 goes to a node with score 2 in the first case, and the corresponding arc goes to a node with score 3 in the second. In Fig. 6, the three nodes having score 2 form a transitive triple in one case, and a cyclic triple in the other. Thus, the two tournaments in each figure are non-isomorphic. That the proper subtournaments are pairwise-isomorphic in each case is a matter of routine verification. We note that the tournaments in each pair are converses of each other.

FIG. 5. Order 6.

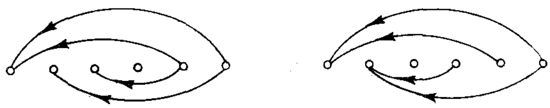

FIG. 6. Order 6.

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2. J. W. MOON, *Topics on Tournaments*, Holt, Rinehart, & Winston, New York, 1968.